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AN ASYMPTOTIC ANALYSIS OF LARGE DEFLECTIONS AND ROTATIONS OF ELASTIC RODS

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Abstract—The distortions within beams and rods undergoing large displacements and rotations are here derived from three-dimensional elasticity by an asymptotic procedure. This procedure, based on the premise that strains vary more gradually along the rod than in transverse directions, takes full account of the shape of the cross section, the traction conditions on the lateral boundary and of any material anisotropy. The manner in which it generates a hierarchy of sets of rod equations is outlined. In the fundamental set, the distortions over each cross section are the anti-clastic curvature and warping associated with the St.-Venant semi-inverse solutions for bending and torsion, suitably corrected to allow for large rotations. The corresponding equations governing the rod configuration are Kirchhoff's equations, with bending and torsional rigidities computed from the St.-Venant distortions. The procedure gives some three-dimensional substance to elastica theory, relating the constitutive assumptions to three-dimensional elasticity. It gives also a logical procedure for obtaining higher order corrections to the theory, and shows how St-Venant's hypothesis concerning details of end loading arises naturally.

1. INTRODUCTION

In an attempt to derive useful information concerning elastic deformations involving large displacements, numerous authors have turned their attention to deformations of rod and beams. These are considered both for their practical importance, and because well established approximate theories are available[1-3], based on linear elasticity but presumed to apply to large displacements.

Many approaches exist. Besides the essentially approximate methods such as elementary beam theory, in which plausible relationships between curvatures and resultant moments are postulated, there are methods involving expansion in powers of the lateral coordinates[4]. These series must be truncated—a procedure whose physical, as opposed to mathematical, implications are not always apparent. For this reason, asymptotic descriptions[5] of rods are to be preferred even if in some cases they generate identical equations.

The aim of any theory of rods or beams is to characterize the deformed configuration of a slender three-dimensional body by a single curve (the *curve of centres*) and certain parameters recording material orientation relative to that curve. The three-dimensional elastic constitutive law is replaced by expressions for resultant forces, moments and generalized moments in terms of extension, curvatures, torsion and the remaining parameters. Each resulting theory must necessarily be approximate, although its accuracy should increase as the representative scale of distance along the axis of the rod increases relative to a typical diameter of the cross section. The present approach takes this requirement as paramount, and postulates that the distribution of deformation gradient over each cross section varies only gradually with axial distance. The variation over each cross section is not specified in advance, but is found by the solution of a sequence of two-dimensional problems which can fully take into account any material anisotropy and inhomogeneity, as well as the traction condition over the lateral boundary. The resulting theory describes deformations of rods and beams suffering small curvature and torsion, but *finite* displacements and rotations.

In the basic approximation, the equations describing the curve of centres are formally equivalent to Kirchhoff's equations[1] for an initially straight rod, with resultant moments linearly related to the curvatures and the torsion. Moreover, it is shown as a logical step in the iteration procedure that for a wide class of materials, including isotropic materials, the bending and torsional rigidities required by *elastica* theory are those appropriate to the St.-Venant semi-inverse solutions for pure bending and pure torsion. Although this has been widely held to be the case, the author is aware of no consistent generation of the result. Thus, this basic approximation gives some three-dimensional substance to elastica theory, and shows how the distortion of any cross section from the plane normal to the curve of centres is described to first approximation by St.-Venant's solutions. This generalizes Ericksen's recent result[6] concerning thin rods in helical configurations. In such configurations, the curvatures and the torsion are constant along the curve of centres, and Ericksen shows that small strain theory has exact solutions in which the distortions are given by St.-Venant's solutions suitably corrected to account for finite rotations. More generally, when the curvatures and the torsion vary gradually along the rod the distortion is given approximately by the corresponding St.-Venant solutions. This idea is partly incorporated in a recent paper[7] by Hegemier and Nair.

Continuation of the iteration procedure will yield theories of successively higher orders, and moreover will generate a procedure for relating the coefficients in such "director theories" to three-dimensional elasticity. It also highlights the fact that a rod theory is only an "outer" approximation in some asymptotic scheme—and that the boundary layer solutions required at the ends of a rod are solutions consistent with St.-Venant's hypothesis.

2. REPRESENTATION OF THE CONFIGURATIONS

We consider elastic deformations of a rod, beam or tube defined by the region $\mathbf{X} \in D \times (0, L)$ of material coordinates X_1, X_2, X_3. In the unstressed reference configuration $\mathbf{X}$ coincides with cartesian coordinates, and D denotes a connected region of the X_1, X_2 plane. Without loss of generality we scale coordinates so that a typical diameter of D has unit length. Then L is large since the rod is slender. A deformation is described by $\mathbf{X} \rightarrow \mathbf{x}(\mathbf{X})$, where $\mathbf{x}$ is the vector of current cartesian coordinates. The deformation gradient $\mathbf{p} = \partial \mathbf{x}/\partial \mathbf{X}$ has components† $p_{ij} = \partial x_i/\partial X_j$, whilst the Cauchy–Green strain tensor is $\mathbf{G} = \mathbf{p}^T\mathbf{p}$. The stored energy density may be written as $EW(\mathbf{p}, \mathbf{X})$, where $W(\mathbf{Hp}, \mathbf{X}) = W(\mathbf{p}, \mathbf{X})$ for all proper orthogonal matrices $\mathbf{H}$, and the explicit dependence on $\mathbf{X}$ allows for material inhomogeneity. Alternatively the energy density may be written as $E\hat{W}(\mathbf{G}, \mathbf{X})$. In either choice E is a characteristic elastic modulus having the dimensions of stress, so that the Piola–Kirchhoff (engineering) stress is

$$E\mathbf{T} = E\frac{\partial W}{\partial \mathbf{p}} \tag{1}$$

where the components $T_{ij} = \partial W/\partial p_{ij}$ of $\mathbf{T}$ are non-dimensional.

The equilibrium equations are

$$\frac{\partial T_{ij}}{\partial X_j} + \frac{\rho_0}{E} f_i = 0 \quad \text{in} \quad D \times (0, L), \tag{2}$$

where ρ_0 is the density in the reference state and $\mathbf{f}$ is the body force per unit mass. The lateral boundary conditions are

$$T_{i\alpha}N_\alpha = E^{-1}g_i \quad \text{over} \quad \partial D \times (0, L) \tag{3}$$

with $\mathbf{g}$ the traction per unit undeformed area and $\mathbf{N}$ the unit outward normal to ∂D, whilst the boundary conditions over the ends $X_3 = 0$, L are for the present left unspecified.

When W is independent of X_3 and $\mathbf{f}$, $\mathbf{g}$ both vanish, eqns (2), (3) possess solutions with $\partial \mathbf{G}/\partial X_3 = 0$, so that $\mathbf{G} = \mathbf{G}(X_\alpha)$. These are helical solutions in which each cross section $X_3 = \text{constant}$ has a similar distorted shape related to two parameters specifying the helix into which the reference fibre $X_3 = 0$ is deformed.‡ For suitably small $E^{-1}\rho_0\mathbf{f}$ and $E^{-1}\mathbf{g}$ we anticipate that deformations are small perturbations from, and gradual modulations to, such helical deformations. An appropriate analytical procedure will be presented in a subsequent paper. However, in many practical situations, although the orientation may vary appreciably along the length of a rod, the curvature and torsion are small quantities. Consequently each distorted cross section approximates to one in which not only $\mathbf{G}$, but also the deformation gradient $\mathbf{p}$, is independent of X_3. Thus, we look for deformations in which $\mathbf{p}$ varies with X_3 only through dependence on the "long scale" variable $Y = \epsilon X_3$, for some small parameter ϵ. To represent

†Throughout this paper Latin indices range over the values 1, 2, 3, whilst Greek indices range over the values 1, 2. Summation convention over repeated indices is used.

‡See Ericksen[6]. Such helical solutions exist also for some distributions of body force and surface traction which, relative to each cross section, are independent of X_3.

such deformations we may write

$$\mathbf{x} = \mathbf{r}(X_3) + \mathbf{H}(Y)\mathbf{u}(X_\alpha, Y), \quad \mathbf{H}^T\mathbf{H} = \mathbf{I}, \quad Y = \epsilon X_3, \tag{4}$$

with $\mathbf{u}(0, Y) = \mathbf{0}$. The curve $\mathbf{x} = \mathbf{r}(X_3)$ then represents the deformed shape of the reference fibre $X_\alpha = 0$. If it has unit tangent $\mathbf{e} = \mathbf{e}(Y)$ and stretch $a = a(Y) = |\mathrm{d}\mathbf{r}/\mathrm{d}X_3|$ which vary only on the scale of Y, we have

$$\mathbf{r}'(X_3) = a(Y)\mathbf{e}(Y) = \mathbf{H}(Y)\mathbf{m}(Y) \quad \text{where} \quad \mathbf{m} = (0, 0, a)^T$$

when $\mathbf{e}$ is taken to be the third column $\mathbf{h}_{*3}$ of $\mathbf{H}$. This, together with $\mathbf{e}_1 = \mathbf{h}_{*1}$ and $\mathbf{e}_2 = \mathbf{h}_{*2}$ forms a right-handed triad of unit vectors which rotates with Y so that

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}X_3} = \epsilon\mathbf{H}'(Y) = \epsilon\mathbf{H}(Y)\hat{\mathbf{\Omega}}(Y), \qquad \hat{\mathbf{\Omega}} + \hat{\mathbf{\Omega}}^T = \mathbf{0} \tag{5}$$

for some skew-symmetric matrix $\hat{\mathbf{\Omega}}$. We then identify $u_i(X_\alpha, Y)$ as cartesian coordinates of points of a distorted cross section relative to the triad $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}$ at corresponding Y (see Fig. 1). Consequently the deformation gradient $\mathbf{p}$ has components†

$$p_{i\alpha} = H_{ij}u_{j,\alpha}\ , \quad p_{i3} = H_{ij}(m_j + \epsilon\hat{\Omega}_{jk}u_k + \epsilon u_{j,Y}) = aH_{i3} + \epsilon H_{ij}(\hat{\Omega}_{jk}u_k + u_{j,Y})$$

from which the rotation $\mathbf{H}$ may be factored out as

$$\mathbf{p} = \mathbf{Hq}, \qquad q_{i\alpha} = u_{i,\alpha}\ , \qquad q_{i3} = a\delta_{i3} + \epsilon(\hat{\Omega}_{ij}u_j + u_{i,Y}). \tag{6}$$

In deformations (4)–(6), the orthogonal matrix $\mathbf{H}$ varies only on the scale of $Y = \epsilon X_3$, so that the curvatures and torsion are small. Moreover they vary only gradually. Thus the parameter ϵ typifies a curvature or torsion of the rod, and ϵ^{-1} is a length over which orientation varies significantly. A suitable numerical value for ϵ will depend on many features of the loading of the rod. However, it is clear from (2) and (3) that deformations can have the form (6) only when the resultant of the loadings $E^{-1}\rho_0\mathbf{f}$ and $E^{-1}\mathbf{g}$ over any cross section are at most $O(\epsilon)$. In deformations largely determined by body forces and lateral surface tractions (e.g. beam bending) this means that ϵ may be chosen as a typical value of (resultant load)/E. For deformations determined essentially by end loadings (e.g. a strut under compression), ϵ is more simply chosen as a typical curvature, torsion or strain.

With little loss of generality we set

$$E^{-1}\rho_0\mathbf{f} = \epsilon\hat{\mathbf{f}}(X_\alpha, Y), \qquad E^{-1}\mathbf{g} = \epsilon\hat{\mathbf{g}}(X_\alpha, Y),$$

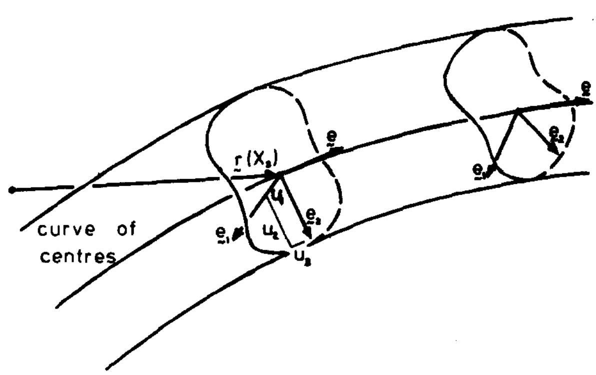

Fig. 1. The orthogonal triad $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}$ at typical cross sections X_3, showing the coordinates u_i of position relative to the point on the curve of centres.

†A comma preceding a subscript denotes partial differentiation with respect to the variable indicated.

so excluding only those situations in which pointwise external loads greatly exceed the average loads on any cross section. Then (2) and (3) become

$$\frac{\partial T_{ij}}{\partial X_j} = -\epsilon \hat{f}_i(X_\alpha, Y) \quad \text{in} \quad D \times (0, L), \tag{7}$$

$$T_{i\alpha} N_\alpha = \epsilon \hat{g}_i(X_\alpha, Y) \quad \text{over} \quad \partial D \times (0, L). \tag{8}$$

Since $W(\mathbf{p}, \mathbf{X}) = W(\mathbf{q}, \mathbf{X})$, the stress $\mathbf{T}$ may be written as $\mathbf{T} = \mathbf{H}\boldsymbol{\tau}$, where

$$\frac{\partial W}{\partial \mathbf{p}} = \mathbf{T} = \mathbf{H}\boldsymbol{\tau} = \mathbf{H}\frac{\partial W}{\partial \mathbf{q}} \quad \text{so that} \quad \boldsymbol{\tau} = \frac{\partial W}{\partial \mathbf{q}}. \tag{9}$$

Thus $\boldsymbol{\tau} = \boldsymbol{\tau}(X_\alpha, Y)$ for homogeneous materials. Also we may allow inhomogeneous materials provided that any explicit dependence of $W(\mathbf{p}, \mathbf{X})$ on X_3 may be written as dependence on the long scale variable $Y = \epsilon X_3$. In terms of $\boldsymbol{\tau}$ eqns (7) and (8) become

$$\frac{\partial \tau_{i\alpha}}{\partial X_\alpha} + \epsilon\left(\hat{\Omega}_{ij}\tau_{j3} + \frac{\partial \tau_{i3}}{\partial Y}\right) = -\epsilon H_{ji}\hat{f}_j \quad \text{in} \quad D \times (0, L) \tag{10}$$

$$\tau_{i\alpha} N_\alpha = \epsilon H_{ji}\hat{g}_j \quad \text{over } \partial D \times (0, L). \tag{11}$$

It is clear that the limiting forms $\epsilon \to 0$ of (6), (9)–(11) describe "cylindrical" deformations $\mathbf{q} = \mathbf{q}(X_\alpha)$ (plane strain, plus axial extension and shear) compatible with zero body force and vanishing tractions over the lateral boundary. We shall use these to give approximations to the distortion $\mathbf{u}(X_\alpha, Y)$ of each cross section.

3. THE PERTURBATION ANALYSIS

Let $\mathbf{q} = \mathbf{q}^*$, $\mathbf{u} = \mathbf{u}^*$ be solutions of the limiting form $\epsilon \to 0$ of eqns (6), (9)–(11), namely

$$\overset{*}{q}_{i\alpha} = \overset{*}{u}_{i,\alpha}, \quad \overset{*}{q}_{i3} = a(Y)\delta_{i3}, \quad \boldsymbol{\tau}^* = \frac{\partial W}{\partial \mathbf{q}^*}, \quad W = W(\mathbf{q}^*, X_\alpha, Y)$$

$$\overset{*}{\tau}_{i\alpha,\alpha} = 0 \quad \text{in} \quad D \times (0, L), \qquad \overset{*}{\tau}_{i\alpha} N_\alpha = 0 \quad \text{over} \quad \partial D \times (0, L). \tag{12}$$

Since (12) contains no derivatives with respect to Y, any Y dependence may be treated as dependence on a parameter. Also, since

$$\mathbf{x}_{,3} = \mathbf{H}(Y)\mathbf{m}(Y) = a(Y)\mathbf{e}(Y)$$

each solution corresponding to fixed Y describes a deformation by which the cylinder having cross section D deforms into another cylinder having axis parallel to $\mathbf{e}$, when subjected to axial stretch a. Details of $\mathbf{u}^*$, $\mathbf{q}^*$ and $\boldsymbol{\tau}^*$ depend on material anisotropy and inhomogeneity, but it is shown in Section 4 that $\mathbf{u}^*(X_\alpha; Y, a)$ is the solution of a variational problem over the two-dimensional region D, and that the resultant load over each cross section is axial and may be written as

$$\mathbf{e}\iint_D \overset{*}{\tau}_{33}\, \mathrm{d}A = \mathbf{e}F^*(a), \qquad \mathrm{d}A = \mathrm{d}X_1\,\mathrm{d}X_2.$$

Solutions to (6), (9)–(11) may be sought in the form

$$\begin{aligned} &\mathbf{u} = \mathbf{u}^*(X_\alpha; Y) + \epsilon\hat{\mathbf{u}}(X_\alpha; Y), \qquad \mathbf{q} = \mathbf{q}^*(X_\alpha; Y) + \epsilon\hat{\mathbf{q}}(X_\alpha; Y) \\ &\boldsymbol{\tau} = \boldsymbol{\tau}^*(X_\alpha; Y) + \epsilon\hat{\boldsymbol{\tau}}(X_\alpha; Y), \qquad a = a(Y), \quad \hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{\Omega}}(Y), \end{aligned} \tag{13}$$

so that the distortion of each cross section is closely approximated by a solution $\mathbf{u}^*$ of (12) for appropriate Y, appropriate stretch a and rotation $\mathbf{H}$.

By inserting (13) into (10) and (11) and using (12) we find that

$$\hat{\tau}_{i\alpha,\alpha} = -(H_{ji}\hat{f}_j + \hat{\Omega}_{ij}\overset{*}{\tau}_{j3} + \overset{*}{\tau}_{i3,Y}) - \epsilon(\hat{\Omega}_{ij}\hat{\tau}_{j3} + \hat{\tau}_{i3,Y})$$

in $D \times (0, L)$, with boundary conditions

$$\hat{\tau}_{i\alpha}N_\alpha = H_{ji}\hat{g}_j(Y)$$

over $\partial D \times (0, L)$. Also, following Toupin and Bernstein[8], $\boldsymbol{\tau}$ may be expanded as

$$\overset{*}{\tau}_{ij} + \epsilon\hat{\tau}_{ij} = \frac{\partial W}{\partial q_{ij}} = \frac{\partial W}{\partial \overset{*}{q}_{ij}} + \epsilon c_{ijkl}\hat{q}_{kl} + \frac{1}{2}\epsilon^2 c_{ijklmn}\hat{q}_{kl}\hat{q}_{mn} + \cdots,$$

so giving

$$\hat{\tau}_{ij} = c_{ijkl}\hat{q}_{kl} + \frac{1}{2}\epsilon c_{ijklmn}\hat{q}_{kl}\hat{q}_{mn} + \ldots$$

where

$$c_{ijkl} = \frac{\partial^2 W}{\partial \overset{*}{q}_{ij}\partial \overset{*}{q}_{kl}} = c_{klij}, \qquad c_{ijklmn} = \frac{\partial^3 W}{\partial \overset{*}{q}_{ij}\partial \overset{*}{q}_{kl}\partial \overset{*}{q}_{mn}}, \text{etc.}$$

Alternatively $\hat{\boldsymbol{\tau}}$ may be expressed in the form

$$\hat{\tau}_{ij} = c_{ijkl}\hat{q}_{kl} + \epsilon\mathcal{N}_{ij}(\hat{\mathbf{u}}, \mathbf{u}^*, X_\alpha; Y, a) \tag{14}$$

where $\mathcal{N}_{ij}$ is the nonlinear function of $\hat{\mathbf{u}}, \mathbf{u}^*$ and their partial derivatives having the value

$$\mathcal{N}_{ij} = \epsilon^{-2}\{\tau_{ij}(\mathbf{q}^* + \epsilon\hat{\mathbf{q}}) - \overset{*}{\tau}_{ij} - \epsilon c_{ijkl}\hat{q}_{kl}\}.$$

Then, from (6) we have

$$\hat{q}_{i\alpha} = \hat{u}_{i,\alpha}, \qquad \hat{q}_{i3} = \hat{\Omega}_{ij}\overset{*}{u}_j + \overset{*}{u}_{i,Y} + \epsilon(\hat{\Omega}_{ij}\hat{u}_j + \hat{u}_{i,Y}),$$

which may be used with (14) to give

$$\begin{aligned}\frac{\partial}{\partial X_\alpha}(c_{i\alpha k\beta}\hat{u}_{k,\beta}) &= -\frac{\partial}{\partial X_\alpha}\{c_{i\alpha k3}(\hat{\Omega}_{km}\overset{*}{u}_m + \overset{*}{u}_{k,Y})\} - (H_{ji}\hat{f}_j + \hat{\Omega}_{ij}\overset{*}{\tau}_{j3} + \overset{*}{\tau}_{i3,Y}) \\ &\quad - \epsilon\left[\hat{\Omega}_{ij}\hat{\tau}_{j3} + \hat{\tau}_{i3,Y} + \frac{\partial}{\partial X_\alpha}\{\mathcal{N}_{i\alpha} + c_{i\alpha k3}(\hat{\Omega}_{km}\hat{u}_m + \hat{u}_{k,Y})\}\right] \\ &\equiv -Y_i(X_\alpha; Y) \quad \text{in} \quad D \times (0, L),\end{aligned} \tag{15}$$

with boundary conditions

$$\begin{aligned}c_{i\alpha k\beta}\hat{u}_{k,\beta}N_\alpha &= -c_{i\alpha k3}(\hat{\Omega}_{km}\overset{*}{u}_m + \overset{*}{u}_{k,Y})N_\alpha + H_{ji}\hat{g}_j \\ &\quad - \epsilon\{\mathcal{N}_{i\alpha} + c_{i\alpha k3}(\hat{\Omega}_{km}\hat{u}_m + \hat{u}_{k,Y})\}N_\alpha \\ &\equiv Z_i(X_\alpha; Y) \quad \text{over} \quad \partial D \times (0, L).\end{aligned} \tag{16}$$

To analyse the structure of eqns (15) and (16) we make the following observations. Suppose, for the moment, that $\mathbf{Y}$ and $\mathbf{Z}$ are known. Then, like (12), these equations involve no derivatives with respect to Y, and so Y may be treated as a parameter. Moreover, (15) and (16) are the linear elastic equations for small displacements $\mathbf{u}^* \to \mathbf{u}^* + \epsilon\hat{\mathbf{u}}$ superposed on the (possibly inhomogeneous) cylindrical deformation $\mathbf{q}^*(X_\alpha, Y)$, and caused by "fictitious" body forces $\epsilon\mathbf{Y}$ and surface tractions $\epsilon\mathbf{Z}$ which, like $\hat{\mathbf{u}}$, are independent of the axial coordinate X_3. Hence functions $\hat{\mathbf{u}}(X_\alpha, Y)$ may be visualized as displacements in a cylinder in which all cross sections are distorted into congruent surfaces. The deformations are the combinations of plane strain and warping which would result from force distributions $\epsilon\mathbf{Y}$ and $\epsilon\mathbf{Z}$, and consequently exist only when these fictitious forces are in static equilibrium. Specifically, the resultant load over each cross section and the moment about any axis parallel to $\mathbf{e}$ must both vanish, so giving

$$\iint_D \mathbf{Y}\,\mathrm{d}A + \oint_{\partial D} \mathbf{Z}\,\mathrm{d}s = \mathbf{0}, \tag{17}$$

$$\iint_D (u_1^* Y_2 - u_2^* Y_1)\,\mathrm{d}A + \oint_{\partial D} (u_1^* Z_2 - u_2^* Z_1)\,\mathrm{d}s = 0. \tag{18}$$

These give four equations connecting $a(Y)$ and the three independent elements of $\hat{\mathbf{\Omega}}(Y)$ which otherwise would appear to be arbitrary in (15) and (16). Their detailed form follows from substitution for $\mathbf{Y}$ and $\mathbf{Z}$ from (15) and (16), and is

$$H_{ji}\left(\iint_D \hat{f}_j\,\mathrm{d}A + \oint_{\partial D} \hat{g}_j\,\mathrm{d}s\right) + \hat{\Omega}_{ij}\iint_D (\tau_{j3}^* + \epsilon\hat{\tau}_{j3})\,\mathrm{d}A + \frac{\mathrm{d}}{\mathrm{d}Y}\iint_D (\tau_{i3}^* + \epsilon\hat{\tau}_{i3})\,\mathrm{d}A = 0$$

$$\begin{aligned} e_{3ki}\left\{\iint_D u_k^* H_{ji}\hat{f}_j\,\mathrm{d}A + \oint_{\partial D} u_k^* H_{ji}\hat{g}_j\,\mathrm{d}s\right\} &= e_{3ji}\iint_D q_{j\alpha}^* c_{i\alpha k3}(\hat{\Omega}_{km}u_m^* + u_{k,Y}^*)\,\mathrm{d}A \\ &\quad - \iint_D e_{3ki}\{u_k^*(\hat{\Omega}_{ij}\tau_{j3}^* + \tau_{i3,Y}^* + \epsilon\hat{\Omega}_{ij}\hat{\tau}_{j3} + \epsilon\hat{\tau}_{i3,Y}) \\ &\quad - \epsilon q_{k\alpha}^*[\mathcal{N}_{i\alpha} + c_{i\alpha j3}(\hat{\Omega}_{jm}\hat{u}_m + \hat{u}_{j,Y})]\}\,\mathrm{d}A, \end{aligned} \tag{19}$$

where e_{ijk} is the alternator.

The first equation is no more than the equilibrium equation connecting the external load $\epsilon\mathbf{HL}$ and the resultant stresses $\mathbf{HF}$, since it may be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}Y}(H_{ij}F_j) + H_{ij}L_j = 0 \tag{20}$$

where

$$F_i \equiv \iint_D \tau_{i3}\,\mathrm{d}A = e_i F^*(a) + \epsilon\iint_D \hat{\tau}_{i3}\,\mathrm{d}A,$$

$$\epsilon L_i \equiv \epsilon H_{ji}\iint_D \hat{f}_j\,\mathrm{d}A + \epsilon H_{ji}\oint_{\partial D} \hat{g}_j\,\mathrm{d}s.$$

Following Love[1] we call $\mathbf{F}$ the *stress-resultant* and $\mathbf{L}$ the *force-resultant*, the components being resolved along $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}$. Likewise, if we use $\mathbf{K}^*$ and $\epsilon\mathbf{M}^*$ to denote the *stress-couple* and the externally applied *couple-resultant* each calculated in the approximation $\mathbf{u} \simeq \mathbf{u}^*$, and defined by

$$K_i^* \equiv e_{ijk}\iint_D u_j^*\tau_{k3}\,\mathrm{d}A$$

$$\epsilon M_i^* \equiv e_{ijk}\left\{\epsilon\iint_D u_j^* H_{lk}\hat{f}_l\,\mathrm{d}A + \epsilon\oint_{\partial D} u_j^* H_{lk}\hat{g}_l\,\mathrm{d}s\right\},$$

we find that the left-hand side of (19) is a determinant which may be identified as $M_3^*(Y)$. Thus, (19) expresses the equilibrium between M_3^* and certain moments associated with the bending and torsion of the rod.

Equations (19) and (20) are compatibility conditions arising from the system (15) and (16) for $\hat{\mathbf{u}}$. They are *exact* equations applying to the functions $\hat{\mathbf{u}}$, $\mathbf{u}^*$ in (13), and together with $\mathbf{H}'(Y) = \hat{\mathbf{\Omega}}(Y)\mathbf{H}(Y)$ govern variations in the orientation $\mathbf{H}$ and stretch a of the rod, so that the configuration

$$\mathbf{r}(X_3) = \epsilon^{-1}\int^{\epsilon X_3} a(\eta)\mathbf{e}(\eta)\,\mathrm{d}\eta \tag{21}$$

may be found. In practice we solve (19) and (20) only approximately, by inserting approximate expressions for $\hat{\mathbf{u}}$ obtained from (15) and (16) either iteratively or by power series expansion in ϵ.

4. THE REFERENCE DEFORMATIONS $\mathbf{q}^*$

The non-linearly elastic solutions $\mathbf{u}^*$ to (12) are simply classified (see Ericksen[9]). In each deformation satisfying (12) we may write

$$W(\mathbf{q}^*, X_\alpha, Y) \equiv \bar{W}(\mathbf{u}^*_{,\alpha}, X_\alpha; Y, a)$$

and then notice that the governing eqns (12) are the Euler–Lagrange equations and the natural boundary conditions which must be satisfied if the "cross-sectional energy"

$$\bar{E} \equiv \iint_D \bar{W}(\mathbf{u}^*_{,\alpha}, X_\alpha; Y, a)\,\mathrm{d}A \tag{22}$$

is to be stationary subject to specified Y and axial stretch a. Thus, solutions $\mathbf{u}^*$ are functions which make $\bar{E}$ take a stationary value $E^*(Y, a)$, and we anticipate that those functions which minimize $\bar{E}$ describe the preferred distorted shapes of the cross sections.

The corresponding stress-resultant is parallel to $\mathbf{e}$ for all material behaviours $W(\mathbf{q}^*, X_\alpha, Y)$. To show this we use

$$0 = e_{ijk}q^*_{jl}\tau^*_{kl} = e_{ijk}(u^*_{j,\alpha}\tau^*_{k\alpha} + a\delta_{j3}\tau^*_{k3})$$

which follows from the symmetry of the Cauchy stress tensor $(\det \mathbf{p})^{-1}\mathbf{T}\mathbf{p}^T$. By integrating over any unit length $D \times (X, X+1)$ of the rod, and using both the divergence theorem and (12), we obtain

$$ae_{3kl}\iint_D \tau^*_{k3}\,\mathrm{d}A = 0$$

which gives

$$\iint_D \tau^*_{\alpha 3}\,\mathrm{d}A = 0.$$

Consequently the stress-resultant has the form $\mathbf{F}^* = \mathbf{e}F^*(a)$, and moreover is related to E^* (see [6], [9]) by

$$F^*(a) = \iint_D \tau^*_{33}\,\mathrm{d}A = \partial E^*/\partial a. \tag{23}$$

Referring to (20) we see that $F^*(a)$ is $O(1)$ if $\mathbf{L}$ is $O(1)$, and that in this case the rod deforms essentially as a string subjected to loading $\epsilon\mathbf{L}(Y)$. For example, if $\hat{\mathbf{f}}$ is purely gravitational and $\hat{\mathbf{g}}$ vanishes, the rod hangs as an extensible catenary, with the $O(\epsilon)$ term in F_i introducing effects of small stiffness such as those computed in [10]. Consequently F^* and $a-1$ need be treated as $O(1)$ only when the rod behaves essentially as a string with small stiffness. The more interesting static problems of rods and beams arise when $\mathbf{L}$, F^* and $a-1$ are small, so that the stress-couple at any cross section is no less important than the stress-resultant. For this reason we confine attention in the paper to *rod* deformations and choose ϵ so that $\tau = O(\epsilon)$ and $\mathbf{q}-\mathbf{I} = O(\epsilon)$. Thus, by taking $\mathbf{q}^* = \mathbf{I}$ we characterize *rod* deformations as those having distortions which are small deviations from the rigid body displacements which align the coordinate axes along $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}$.

The choice $\mathbf{q}^* = \mathbf{I}$, $\tau^* = \mathbf{0}$ reduces eqn (19) to

$$M_3^* = e_{3\alpha i}\iint_D c_{i\alpha k3}\hat{\Omega}_{k\beta}X_\beta\,\mathrm{d}A + O(\epsilon).$$

Since we expect bending and torsion to be possible without such a large couple-resultant, this implies a preferred choice of axis $X_\alpha = 0$. For simplicity we restrict attention to materials which are homogeneous over each cross section so that $\partial c_{ijkl}/\partial X_\alpha = 0$. Then the $O(1)$ terms disappear from M_3^* if, as is conventional, the reference line $X_\alpha = 0$ is taken as the axis of centroids of the cross sections D. Also, since $\mathbf{L}$ and M_3^* are $O(\epsilon)$ we lose little generality by taking $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$ as $O(\epsilon)$ in the following analysis.

5. ANALYSIS OF THE DISTORTIONS OF THE CROSS SECTIONS

When we take reference deformations with $u_i^* = \delta_{i\beta}X_\beta$ and $q_{33}^* = 1$, we must allow some axial extension in $\epsilon\hat{\mathbf{q}}$. Consequently, we set $a = 1 + \epsilon\hat{a}(Y)$ in (6) and (21) and incorporate a contribution $\hat{a}\delta_{i3}$ into $\hat{q}_{i3}$ in (15) and (16) and all succeeding equations, so that (6) gives

$$\hat{q}_{i\alpha} = \hat{u}_{i,\alpha}, \qquad \hat{q}_{i3} = \hat{a}\delta_{i3} + \hat{\Omega}_{i\beta}X_\beta + \epsilon(\hat{\Omega}_{im}\hat{u}_m + \hat{u}_{i,Y}). \tag{24}$$

Thus, eqns (15) in D become

$$\frac{\partial}{\partial X_\alpha}(c_{i\alpha k\beta}\hat{u}_{k,\beta}) = -\frac{\partial}{\partial X_\alpha}\{c_{i\alpha k3}(\hat{a}\delta_{k3} + \hat{\Omega}_{k\beta}X_\beta)\} - H_{ji}\hat{f}_j - \epsilon\left[\hat{\Omega}_{ij}\hat{\tau}_{j3} + \hat{\tau}_{i3,Y} + \frac{\partial}{\partial X_\alpha}\{\mathcal{N}_{i\alpha} + c_{i\alpha k3}(\hat{\Omega}_{km}\hat{u}_m + \hat{u}_{k,Y})\}\right]. \tag{15'}$$

The boundary conditions over ∂D are obtained similarly from (16) and are

$$c_{i\alpha k\beta}\hat{u}_{k,\beta}N_\alpha = -c_{i\alpha k3}(\hat{a}\delta_{k3} + \hat{\Omega}_{k\beta}X_\beta)N_\alpha + H_{ji}\hat{g}_j - \epsilon\{\mathcal{N}_{i\alpha} + c_{i\alpha k3}(\hat{\Omega}_{km}\hat{u}_m + \hat{u}_{k,Y})\}N_\alpha. \tag{16'}$$

These are to be solved for $\hat{\mathbf{u}}$ and $\hat{\tau}$, which are then used in evaluating the stress-resultants

$$F_j = \epsilon\hat{F}_j \equiv \epsilon\iint_D \hat{\tau}_{j3}\,\mathrm{d}A$$

which occur in (20). Since this reduces to

$$\frac{\mathrm{d}\hat{\mathbf{F}}}{\mathrm{d}Y} + \hat{\mathbf{\Omega}}\hat{\mathbf{F}} = \mathbf{H}^T\frac{\mathrm{d}}{\mathrm{d}Y}(\mathbf{H}\hat{\mathbf{F}}) = -\epsilon^{-1}\mathbf{L},$$

we may introduce the components of $\hat{\mathbf{\Omega}}$ by

$$\hat{\mathbf{\Omega}} = \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} \tag{25}$$

and obtain

$$\frac{\mathrm{d}\hat{F}_1}{\mathrm{d}Y} - \gamma\hat{F}_2 + \beta\hat{F}_3 + \epsilon^{-1}L_1 = 0$$

$$\frac{\mathrm{d}\hat{F}_2}{\mathrm{d}Y} + \gamma\hat{F}_1 - \alpha\hat{F}_3 + \epsilon^{-1}L_2 = 0$$

$$\frac{\mathrm{d}\hat{F}_3}{\mathrm{d}Y} - \beta\hat{F}_1 + \alpha\hat{F}_2 + \epsilon^{-1}L_3 = 0 \tag{26}$$

which, apart from a change of notation, are the equilibrium equations used by Kirchhoff (see [1], p. 387).

In (25) and (26), $\epsilon\alpha/a$ and $\epsilon\beta/a$ are the conventional curvatures κ and κ' whilst $\epsilon\gamma/a$ is the torsion of the rod ($a \simeq 1$). Together these measure the rate at which $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}$ rotate as Y increases.

The aim of this paper is to justify the usual assumptions concerning $\hat{\mathbf{F}}$ by considering (15′) and (16′). Specifically we show that unless the rod behaves essentially as a string or bar in tension or a bar in compression, with $\hat{\mathbf{F}} \simeq \hat{F}_3\mathbf{e} \simeq \hat{a}A\mathbf{e}$, then $\hat{\mathbf{F}}$ is not functionally related to $\hat{\mathbf{\Omega}}$ and $\hat{a}$. It is the first approximation $\epsilon\hat{\mathbf{K}}$ to the stress-couple $\mathbf{K}^*$ which is related to $\hat{\mathbf{\Omega}}$ in the basic approximate constitutive law.

The eqns (15′) and (16′) for $\hat{\mathbf{u}}$ may be split into three parts, each of which corresponds to a self-consistent problem of the type (15) and (16). We write

$$\hat{\mathbf{u}} = \bar{\mathbf{u}} + \epsilon\bar{\bar{\mathbf{u}}} + \epsilon\mathbf{v}, \qquad \mathbf{q} = \mathbf{I} + \epsilon\hat{\mathbf{q}} = \mathbf{I} + \epsilon\bar{\mathbf{q}} + \epsilon^2(\bar{\bar{\mathbf{q}}} + \tilde{\mathbf{q}}), \qquad \boldsymbol{\tau} = \epsilon\hat{\boldsymbol{\tau}} = \epsilon\bar{\boldsymbol{\tau}} + \epsilon^2(\bar{\bar{\boldsymbol{\tau}}} + \tilde{\boldsymbol{\tau}}) \tag{27}$$

where the dominant contribution $\bar{\mathbf{u}}$ to $\hat{\mathbf{u}}$ is a solution of

$$\frac{\partial}{\partial X_\alpha}(c_{i\alpha k\beta}\bar{u}_{k,\beta}) = -\frac{\partial}{\partial X_\alpha}\{c_{i\alpha k3}(\hat{a}\delta_{k3} + \hat{\Omega}_{k\beta}X_\beta)\} \quad \text{in} \quad D$$

$$c_{i\alpha k\beta}\bar{u}_{k,\beta}N_\alpha = -c_{i\alpha k3}(\hat{a}\delta_{k3} + \hat{\Omega}_{k\beta}X_\beta)N_\alpha \quad \text{over} \quad \partial D. \tag{28}$$

In (28), the right hand sides represent a distribution of fictitious force which is self-equilibriating because $X_\alpha = 0$ is chosen as the line of centroids.

Alternatively, since $\bar{q}_{k\beta} = \bar{u}_{k,\beta}$, $\bar{q}_{k3} = \hat{a}\delta_{k3} + \hat{\Omega}_{k\beta}X_\beta$ is the limiting form of (24) as $\epsilon \to 0$, the distortions $\bar{\mathbf{u}}$ corresponding to each choice $\hat{\mathbf{\Omega}}$ and $\hat{a}$ are those predicted for helical configurations by *small strain* theory (Ericksen[6]). They are discussed further in Section 6.

The contributions $\epsilon\bar{\bar{\mathbf{u}}}$ at any Y are those induced directly by the external loadings $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$, which themselves are $O(\epsilon)$. These loadings are not usually self-equilibriating—but have resultant $\mathbf{HL}$ and axial moment $M_3^*\mathbf{e}$. Consequently, if A is the area of D and

$$C \equiv \iint_D (X_1{}^2 + X_2{}^2)\,\mathrm{d}A$$

is the second moment of area about the axis $X_\alpha = 0$ we can obtain a compatible system of equations by writing

$$\frac{\partial}{\partial X_\alpha}(c_{i\alpha k\beta}\bar{\bar{u}}_{k,\beta}) = -\epsilon^{-1}\{H_{ji}\hat{f}_j - A^{-1}L_i\} + \epsilon^{-1}e_{3ji}X_jM_3^*C^{-1} \quad \text{in} \quad D$$

$$c_{i\alpha k\beta}\bar{\bar{u}}_{k,\beta}N_\alpha = \epsilon^{-1}H_{ji}\hat{g}_j \quad \text{over} \quad \partial D. \tag{29}$$

The choice $A^{-1}L_i + e_{3ji}X_jM_3^*C^{-1}$ is just one of many possible fictitious force distributions having the required resultant and moment, but any other sensible choice will cause only $O(\epsilon)$ differences in $\epsilon\bar{\mathbf{u}}$. The functions $\bar{\bar{\mathbf{u}}}$ describe cylindrical distortions which account for the details of the loading distributions. Moreover, they vanish ($\bar{\bar{\mathbf{u}}} = \mathbf{0}$) when the body forces are uniform (like gravitational forces) with $\hat{\mathbf{f}} = A^{-1}\mathbf{HL}$, $M_3^* = 0$ and $\hat{\mathbf{g}} = \mathbf{0}$.

Since (26) shows that

$$L_i = -\epsilon(\hat{\Omega}_{ij}\hat{F}_j + \hat{F}_{i,Y}) = -\epsilon A(\hat{\Omega}_{ij}\langle\hat{\tau}_{j3}\rangle + \langle\hat{\tau}_{j3,Y}\rangle),$$

where $\langle f\rangle$ denotes the mean value over D of any quantity f, the third contribution $\epsilon\mathbf{v}$ is governed by

$$\frac{\partial}{\partial X_\alpha}(c_{i\alpha k\beta}v_{k,\beta}) = -e_{3ji}X_j\epsilon^{-1}M_3^*C^{-1} - \hat{\Omega}_{ij}(\hat{\tau}_{j3} - \langle\hat{\tau}_{j3}\rangle) - (\hat{\tau}_{i3,Y} - \langle\hat{\tau}_{i3,Y}\rangle)$$

$$-\frac{\partial}{\partial X_\alpha}\{\mathcal{N}_{i\alpha} + c_{i\alpha k3}(\hat{\Omega}_{km}\hat{u}_m + \hat{u}_{k,Y})\} \quad \text{in} \quad D \tag{30}$$

$$c_{i\alpha k\beta}v_{k,\beta}N_\alpha = \{\mathcal{N}_{i\alpha} + c_{i\alpha k3}(\hat{\Omega}_{km}\hat{u}_m + \hat{u}_{k,Y})\}N_\alpha \quad \text{over} \quad \partial D.$$

Since (28) and (29), like (15′) and (16′), are self-consistent systems, so also is (30). Unlike (28) and (29), which are linear in $\bar{\mathbf{u}}$ and $\bar{\bar{\mathbf{u}}}$ and correspond to deformation gradients and stresses

$$\bar{q}_{i\alpha} = \bar{u}_{i,\alpha}, \quad \bar{q}_{i3} = \hat{a}\delta_{i3} + \hat{\Omega}_{i\beta}X_\beta; \quad \bar{\tau}_{ij} = c_{ijkl}\bar{q}_{kl}$$
$$\bar{\bar{q}}_{i\alpha} = \bar{\bar{u}}_{i,\alpha}, \quad \bar{\bar{q}}_{i3} = 0; \quad \bar{\bar{\tau}}_{ij} = c_{ijkl}\bar{\bar{q}}_{kl} = c_{ijk\alpha}\bar{\bar{q}}_{k\alpha}, \tag{31}$$

the system (30) is non-linear in $\mathbf{v}$. Moreover it involves the function $\bar{\mathbf{u}}$ and $\bar{\bar{\mathbf{u}}}$, since (14) and (25) imply that

$$\tilde{\tau}_{ij} = c_{ijkl}\tilde{q}_{kl} + \mathcal{N}_{ij}(\hat{\mathbf{u}}), \quad \tilde{q}_{i\alpha} = v_{i,\alpha}, \quad \tilde{q}_{i3} = 0, \quad \hat{\mathbf{u}} = \bar{\mathbf{u}} + \epsilon(\bar{\bar{\mathbf{u}}} + \mathbf{v}). \tag{32}$$

However, $\mathbf{v}$ enters the right hand side of (30) only through terms which are $O(\epsilon)$, so that it appears feasible to construct solutions either by iteration or be expansion in powers of ϵ. The form of the distortions $\hat{\mathbf{u}}$ then becomes clear. Since $\bar{\mathbf{u}}$ is linear in $\hat{a}$ and the three independent elements of $\hat{\Omega}$, the first approximation to $\mathbf{v}$ involves derivatives of these with respect to Y. Each successive iteration introduces a higher derivative, but also a corresponding extra factor of ϵ. Consequently, when $\hat{\tau} = \bar{\tau} + \epsilon(\bar{\bar{\tau}} + \tilde{\tau})$ is substituted into each $\hat{F}_i$ in (26), and also into (19) to give a further scalar equation, the right hand sides involve successively higher derivatives of $\hat{\Omega}$ and $\hat{a}$ multiplied by correspondingly increasing powers of ϵ.

The iteration will not terminate naturally. In elasticity the stress-resultant $\epsilon\hat{\mathbf{F}}$ at one cross section will be affected (perhaps minutely) by $\hat{\Omega}$ and $\hat{a}$ at any other value of Y. However, as $\epsilon \to 0$ the dependence becomes ever more localized in Y. The first approximation $\hat{\mathbf{u}} \simeq \bar{\mathbf{u}}$ of our procedure may then be regarded as the first term in an "outer" expansion for the distortion of cross sections. Subsequent iterates produce terms which give higher order corrections. Correspondingly the rod configuration is obtained as an outer expansion arising from substitution into (26) and (19). The lowest order terms are found to be Kirchhoff's rod equations. Obviously such expansions do not apply near the "ends" of the rod, where "boundary layers" or "transition regions" must occur. These are described by deformations which exhibit St.-Venant's hypothesis, and may be analysed using Toupin's approach [11] (see Section 8).

6. THE DISTORTIONS $\bar{\mathbf{u}}$

To solve (28) for the dominant distortions $\bar{\mathbf{u}}$ we first consider isotropic materials, for which standard results are readily available. Then the linear elastic constitutive law (31) for $\bar{\tau}_{ij}$ becomes

$$\bar{\tau}_{ij} = \lambda\delta_{ij}\bar{q}_{mm} + \mu(\bar{q}_{ij} + \bar{q}_{ji})$$

where λ, μ are the Lamé constants non-dimensionalized with respect to E and $\bar{\tau} = \bar{\tau}^T$ is indistinguishable from a Cauchy stress. Substituting for $\hat{\Omega}$ and $\bar{\mathbf{q}}$ from (25) and (31) we see that (28) reduces to

$$\frac{\partial}{\partial X_1}(\lambda\bar{u}_{\alpha,\alpha} + 2\mu\bar{u}_{1,1}) + \frac{\partial}{\partial X_2}\{\mu(\bar{u}_{1,2} + \bar{u}_{2,1})\} = -\frac{\partial}{\partial X_1}\{\lambda(-\beta X_1 + \alpha X_2 + \hat{a})\}$$

$$\frac{\partial}{\partial X_1}\{\mu(\bar{u}_{1,2} + \bar{u}_{2,1})\} + \frac{\partial}{\partial X_2}(\lambda\bar{u}_{\alpha,\alpha} + 2\mu\bar{u}_{2,2}) = -\frac{\partial}{\partial X_2}\{\lambda(-\beta X_1 + \alpha X_2 + \hat{a})\}$$

$$\frac{\partial}{\partial X_1}(\mu \bar{u}_{3,1})+\frac{\partial}{\partial X_2}(\mu \bar{u}_{3,2})=0,$$

with appropriate boundary conditions.

Solutions are unique when the conditions $\bar{\mathbf{u}}=\mathbf{0}$, $\bar{u}_{1,2}=\bar{u}_{2,1}$ are imposed at $X_\alpha=0$ to eliminate arbitrary translations and a rigid rotation about the cylindrical axis. We note that $\bar{u}_3$ does not occur in the first two equations, and has the form $\bar{u}_3=\gamma\phi(X_\alpha)$ where the harmonic function ϕ is the standard warping function for the cross section D.

To analyse the distortions fully we find it advantageous to introduce the Prandtl stress functions $\theta_i(X_\alpha, Y)$ for which

$$\bar{\tau}_{i\alpha}=c_{i\alpha k\beta}\bar{u}_{k,\beta}+c_{i\alpha k3}(\hat{a}\delta_{k3}+\hat{\Omega}_{k\beta}X_\beta)=e_{\alpha\beta}\theta_{i,\beta} \quad \text{in} \quad D, \tag{33}$$

where $e_{\alpha\beta}$ is the alternator. Without loss of generality, the boundary conditions may be taken as

$$\boldsymbol{\theta}=\mathbf{0} \quad \text{on} \quad \partial D. \tag{34}$$

For isotropic materials the definitions of $\bar{\tau}_{3\alpha}$ then give

$$\mu(\bar{u}_{3,1}-\gamma X_2)=\theta_{3,2}, \qquad \mu(\bar{u}_{3,2}+\gamma X_1)=-\theta_{3,1}$$

from which $\bar{u}_3$ may be eliminated to give

$$\nabla_2^2\theta_3=-2\mu\gamma \quad \text{in} \quad D, \qquad \theta_3=0 \quad \text{on} \quad \partial D. \tag{35}$$

Hence $\theta_3=\mu\gamma\Psi(X_\alpha)$, where Ψ is the classical torsion function for D. The corresponding warping is $\bar{u}_3=\gamma\phi(X_\alpha)$ where ϕ is the harmonic conjugate of $\Psi+\frac{1}{2}(X_1^2+X_2^2)$.

The remaining equations are

$$\lambda\bar{u}_{\alpha,\alpha}+2\mu\bar{u}_{1,1}+\lambda(-\beta X_1+\alpha X_2+\hat{a})=\theta_{1,2}$$
$$\mu(\bar{u}_{1,2}+\bar{u}_{2,1})=-\theta_{1,1}=\theta_{2,2}$$
$$\lambda\bar{u}_{\alpha,\alpha}+2\mu\bar{u}_{2,2}+\lambda(-\beta X_1+\alpha X_2+\hat{a})=-\theta_{2,1}.$$

When the Airy stress function $\chi(X_\alpha)$ is introduced so that $\theta_1=\chi_{,2}$, $\theta_2=-\chi_{,1}$ and the $\bar{u}_{\alpha,\beta}$ are eliminated, we obtain $\nabla_2^4\chi=0$ in D with $\chi=0$, $\partial\chi/\partial n=0$ on ∂D. Consequently, $\theta_\alpha=0$ throughout D so that

$$\bar{u}_{1,1}=\bar{u}_{2,2}=\sigma(\beta X_1-\alpha X_2-\hat{a})=-\sigma\bar{q}_{33}, \qquad \bar{u}_{1,2}=-\bar{u}_{2,1}, \tag{36}$$

and the distortions satisfying $\bar{u}_\alpha=0$, $\bar{u}_{1,2}=\bar{u}_{2,1}$ on $X_\alpha=0$ are

$$\bar{u}_1=\sigma\left\{-\alpha X_1X_2+\frac{1}{2}\beta(X_1^2-X_2^2)-\hat{a}X_1\right\}$$
$$\bar{u}_2=\sigma\left\{\frac{1}{2}\alpha(X_1^2-X_2^2)+\beta X_1X_2-\hat{a}X_2\right\}$$

where $\sigma\equiv\frac{1}{2}\lambda(\lambda+\mu)^{-1}$ is Poisson's ratio. Thus we find that the distortions $\bar{\mathbf{u}}$ at *any* cross section are the linear combinations

$$\bar{\mathbf{u}}(X_\alpha, Y)=\alpha\mathbf{w}^{(1)}(X_\alpha)+\beta\mathbf{w}^{(2)}(X_\alpha)+\gamma\boldsymbol{\phi}(X_\alpha)+\hat{a}\mathbf{c}(X_\alpha) \tag{37}$$

of the displacements $\mathbf{c}$, $\boldsymbol{\phi}$, $\mathbf{w}^{(\alpha)}$ obtained from St.-Venant's semi-inverse solutions for extension, torsion and bending. The extension $\epsilon\hat{a}$, twist $\epsilon\gamma$ and curvatures $\epsilon\alpha$, $\epsilon\beta$ associated with bending about the X_1 and X_2 axes may vary with Y. They are constants in the special case of helical configurations discussed by Ericksen[6].

The preceding analysis is readily modified to apply to any anisotropic material for which planes $X_3 = $ constant are planes of structural symmetry. For these materials the value of W is unchanged when $\bar{q}_{\alpha 3}$ is replaced by $-\bar{q}_{\alpha 3}$, and all elastic moduli c_{ijkl} which contain the index 3 an odd number of times are zero. Also, since $\bar{\tau}_{ij} = c_{ijkl}\bar{q}_{kl}$ is a linear elastic constitutive law, $\bar{\tau}$ is symmetric so that $c_{ijkl} = c_{jikl} = c_{klij}$. Although such solutions may be found in Love[1] (p. 345) and Ericksen[6], the present author believes that use of the stress functions θ_i better reveals their structure.

In uniaxial tension with extension $\bar{q}_{33} = \bar{a}$, the stresses are

$$\bar{\tau}_{\alpha\beta} = 0, \qquad \bar{\tau}_{\alpha 3} = \bar{\tau}_{3\alpha} = 0, \qquad \bar{\tau}_{33} = \bar{a}$$

when E is chosen as the Young's modulus for uniaxial stress. The general corresponding deformation gradients are

$$\bar{q}_{\alpha\beta} = \bar{a}K_{\alpha\beta} + \delta e_{\alpha\beta}, \qquad \bar{q}_{\alpha 3} = \bar{q}_{3\alpha} = 0, \qquad \bar{q}_{33} = \bar{a}$$

where $K_{\alpha\beta} = K_{\beta\alpha}$. The arbitrary parameter δ allows for rotations about the X_3 axis, whilst the symmetric matrix $\mathbf{K}$ describes the Poisson contractions. We notice that in isotropic materials for which $K_{\alpha\beta} = -\sigma\delta_{\alpha\beta}$, the solutions (36) have the form $\bar{q}_{\alpha\beta} = -\sigma\bar{q}_{33}\delta_{\alpha\beta} + e_{\alpha\beta}p(X_\alpha)$. Thus, each fibre $X_\alpha = $ constant undergoes Poisson contraction appropriate to $\bar{\tau}_{\alpha\beta} = 0$, but undergoes a rotation p sufficient to satisfy the compatibility conditions $\bar{q}_{\alpha\beta,\gamma} = \bar{q}_{\alpha\gamma,\beta}$.

We seek similar solutions $\theta_\alpha \equiv 0$ to (28) in the present case, with $\bar{q}_{13} = -\gamma X_2$, $\bar{q}_{23} = \gamma X_1$, $\bar{q}_{33} = \hat{a} - \beta X_1 + \alpha X_2$ as before, and so propose that

$$\bar{q}_{\alpha\beta} = \bar{q}_{33}K_{\alpha\beta} + e_{\alpha\beta}p(X_1, X_2). \tag{38}$$

It is then easily verified by substitution into (31) that $\bar{\tau}_{\alpha\beta} = 0$, so that the assumed form (38) is compatible with $\theta_\alpha = 0$ for all choices of $p(X_1, X_2)$.

Using $\bar{q}_{33,1} = -\beta$ and $\bar{q}_{33,2} = \alpha$, we find that the remaining compatibility conditions are satisfied when

$$\begin{aligned} p &= (\alpha K_{11} + \beta K_{12})X_1 + (\alpha K_{21} + \beta K_{22})X_2, \\ \bar{u}_1 &= \alpha(K_{11}X_1X_2 + K_{12}X_2^2) - \frac{1}{2}\beta(K_{11}X_1^2 - K_{22}X_2^2) + \hat{a}K_{1\alpha}X_\alpha, \\ \bar{u}_2 &= -\frac{1}{2}\alpha(K_{11}X_1^2 - K_{22}X_2^2) - \beta(K_{12}X_1^2 + K_{22}X_1X_2) + \hat{a}K_{2\alpha}X_\alpha. \end{aligned} \tag{39}$$

The distortions (39) describe anticlastic curvature and Poisson contraction, and are independent of the torsion parameter γ. Thus the torsion solution is a pure warping with $\bar{u}_\alpha = 0$, $\bar{u}_3 \propto \gamma$.

The stresses $\bar{\tau}_{3\alpha} = \bar{\tau}_{\alpha 3}$ reduce to the forms

$$\bar{\tau}_{3\alpha} = \bar{\tau}_{\alpha 3} = c_{3\alpha 3\beta}\bar{q}_{3\beta} + c_{3\alpha\beta 3}\bar{q}_{\beta 3} = L_{\alpha\beta}(\bar{u}_{3,\beta} - \gamma e_{\beta\delta}X_\delta),$$

where $L_{\alpha\beta} = L_{\beta\alpha} = c_{3\alpha 3\beta}$ are the shear moduli used by Ericksen[6]. Since $\bar{\tau}_{3\alpha} = e_{\alpha\beta}\theta_{3,\beta}$, the stress function θ_3 is determined from

$$\begin{pmatrix} \theta_{3,2} \\ -\theta_{3,1} \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \bar{u}_{3,1} - \gamma X_2 \\ \bar{u}_{3,2} + \gamma X_1 \end{pmatrix}$$

in D, with $\theta_3 = 0$ on ∂D. When we write $\theta_3 = \gamma\psi(X_\alpha)$ and eliminate $\bar{u}_3$ we find that $\psi(X_\alpha)$ satisfies

$$\begin{aligned} L_{\alpha\beta}\psi_{,\alpha\beta} \equiv L_{11}\psi_{,11} + 2L_{12}\psi_{,12} + L_{22}\psi_{,22} &= -2\Delta \quad \text{in} \quad D \\ \psi &= 0 \quad \text{on} \quad \partial D \end{aligned} \tag{40}$$

where $\Delta \equiv L_{11}L_{22} - L_{12}L_{21}$. The corresponding warping function $\phi(X_\alpha)$, for which $\bar{u}_3 = \gamma\phi$, is then readily found from

$$\phi_{,1} = X_2 + \Delta^{-1} L_{2\alpha}\psi_{,\alpha}, \qquad \phi_{,2} = -X_1 - \Delta^{-1} L_{1\alpha}\psi_{,\alpha} .$$

After suitable rotation of axes and rescaling of coordinates, the eqn (40) may be put into canonical form $\psi_{,XX} + \psi_{,YY} = -2$. Consequently, the stresses $\bar{\tau}_{\alpha 3}$ and warping $\phi(X_\alpha)$ are simply related to those of an isotropic rod having cross section which is an affine stretching of D.

For materials having more general anisotropy the equations for θ_1, θ_2 and θ_3 will not generally allow solutions $\theta_\alpha \equiv 0$. However, distortions will still have the form (37) with the bending and torsion functions $\mathbf{w}^{(\alpha)}$, ϕ determined as solutions of the three coupled second-order equations (28) for $\bar{\mathbf{u}}$. These equations have constant right hand sides, since $\bar{q}_{k3}$ is linear in X_1 and X_2, but displacements $\bar{u}_\alpha$ will not generally be quadratic in X_1 and X_2. To $O(\epsilon)$, the stress-resultant $\epsilon\mathbf{F}$ and stress-couple $\epsilon\hat{\mathbf{K}}$ will be linear in $\hat{a}$ and the components of $\hat{\mathbf{\Omega}}$.

7. DERIVATION OF THE ROD EQUATIONS

For the materials considered in Section 6, with distortions given by (39) and (40), the stress resultant is $\mathbf{F} \simeq \epsilon\mathbf{F}$, where

$$\bar{F}_\alpha \equiv \iint_D \bar{\tau}_{\alpha 3}\, \mathrm{d}A = \gamma \iint_D e_{\alpha\beta}\phi_{,\beta}\, \mathrm{d}A = 0$$

$$\bar{F}_3 \equiv \iint_D \bar{\tau}_{33}\, \mathrm{d}A = \iint_D (c_{33\alpha\beta}\bar{q}_{\alpha\beta} + c_{3333}\bar{q}_{33})\, \mathrm{d}A$$

$$= \iint_D (c_{33\alpha\beta}K_{\alpha\beta} + c_{3333})\bar{q}_{33}\, \mathrm{d}A = \iint_D (\hat{a} - \beta X_1 + \alpha X_2)\, \mathrm{d}A$$

$$= A\hat{a}.$$

(Recall that stresses have been non-dimensionalized with respect to the Young's modulus E.) Reference to (26) then shows that if $\mathbf{L} \neq O(\epsilon)$ the rod behaves like a string or bar undergoing small extension. In such cases the configuration is determined essentially by

$$\frac{\mathrm{d}}{\mathrm{d}Y}(\hat{F}_3\mathbf{e}) + \epsilon^{-1}\mathbf{L} = \mathbf{0}, \qquad \mathbf{e}\cdot\mathbf{e} = 1, \qquad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}Y} = \epsilon^{-1}\mathbf{e}(Y),$$

which follow from the approximations $\hat{F}_\alpha \ll \hat{F}_3$, $a \simeq 1$. To this approximation $\hat{F}_3$ arises as a reaction to the constraint $\mathbf{e}\cdot\mathbf{e} = 1$, and causes an extension $\epsilon\hat{a} \simeq A^{-1}\epsilon\hat{F}_3$. These equations are the basis of theories for an extensible string and for bars and columns under compression.

More general deformations of rods correspond to $\mathbf{L} = 0(\epsilon^2)$ and $\epsilon\hat{a} = 0(\epsilon^2)$, so that all components of $\mathbf{F}$ are $O(\epsilon^2)$ even though $\tau = O(\epsilon)$. This might suggest that, to determine rod deformations, we need find $\bar{\bar{\tau}}$ and at least one iterate for $\bar{\tau}$ in order to evaluate non-vanishing approximations to $\hat{\mathbf{F}}$ for substitution into (26). However, standard considerations of equilibrium and standard treatments of beam bending indicate that such calculations may be avoided, and that the stress-resultant $\epsilon\hat{\mathbf{F}}$ is related to the exact expressions

$$M_i = \epsilon\hat{M}_i \equiv e_{ijk}\left\{\epsilon \iint_D u_j H_{lk}\hat{f}_l\, \mathrm{d}A + \epsilon \oint_{\partial D} u_j H_{lk}\hat{g}_l\, \mathrm{d}s\right\}$$

$$K_i = \epsilon\hat{K}_i \equiv e_{ijk} \iint_D \epsilon u_j \hat{\tau}_{k3}\, \mathrm{d}A \qquad (41)$$

for the couple-resultant $\mathbf{M}$ and the stress-couple $\mathbf{K}$. We shall now show how approximations to this relationship are generated naturally, and how, in the first approximation, $\epsilon\hat{\mathbf{F}}$ is not determined by any constitutive law but is connected to the approximations $\epsilon\mathbf{M}^*$, $\epsilon\bar{\mathbf{K}}$ obtained from (41) on the basis of $\mathbf{u} \simeq \mathbf{u}^*$, $\hat{\tau}_{k3} \simeq \bar{\tau}_{k3}$.

We exploit the symmetry of $\tau\mathbf{q}^T$ which is embodied in (1), (14) and (27), and is not an extra restriction. Substituting from (27) we recover $\bar{\tau}_{ij} = \bar{\tau}_{ji}$ and obtain the additional symmetry condition

$$\bar{\bar{\tau}}_{ij} + \tilde{\tau}_{ij} + \bar{\tau}_{il}\bar{q}_{jl} = \bar{\bar{\tau}}_{ji} + \tilde{\tau}_{ji} + \bar{\tau}_{jl}\bar{q}_{il} + O(\epsilon).$$

Since $\bar{\bar{\tau}}_{ij} = c_{ijkl}\bar{\bar{q}}_{kl} = c_{jikl}\bar{\bar{q}}_{kl} = \bar{\bar{\tau}}_{ji}$ the shear resultants are

$$\begin{aligned}\epsilon\hat{F}_\alpha &\equiv \iint_D \epsilon\hat{\tau}_{\alpha 3}\,\mathrm{d}A = \epsilon^2 \iint_D (\bar{\bar{\tau}}_{\alpha 3} + \tilde{\tau}_{\alpha 3})\,\mathrm{d}A \\ &= \epsilon^2 \iint_D \{\bar{\bar{\tau}}_{3\alpha} + \tilde{\tau}_{3\alpha} + \bar{\tau}_{3l}\bar{q}_{\alpha l} - \bar{\tau}_{\alpha l}\bar{q}_{3l} + O(\epsilon)\}\,\mathrm{d}A. \end{aligned} \tag{42}$$

Using (29) and (31) we may evaluate the first term as

$$\begin{aligned}\epsilon^2 \iint_D \bar{\bar{\tau}}_{3\alpha}\,\mathrm{d}A &= \epsilon^2 \oint_{\partial D} X_\alpha \bar{\bar{\tau}}_{3\beta} N_\beta\,\mathrm{d}s - \epsilon^2 \iint_D X_\alpha \frac{\partial \bar{\bar{\tau}}_{3\beta}}{\partial X_\beta}\,\mathrm{d}A \\ &= \epsilon \oint_{\partial D} X_\alpha H_{j3}\hat{g}_j\,\mathrm{d}s + \epsilon \iint_D X_\alpha (H_{j3}\hat{f}_j - A^{-1}L_3)\,\mathrm{d}A = \epsilon e_{\beta\alpha} M^*_\beta. \end{aligned}$$

Performing similar manipulations on the second term of (42) we obtain

$$\begin{aligned}\epsilon^2 \iint_D \tilde{\tau}_{3\alpha}\,\mathrm{d}A &= \epsilon^2 \iint_D X_\alpha \{\hat{\Omega}_{3j}(\hat{\tau}_{j3} - \langle \hat{\tau}_{j3}\rangle) + \hat{\tau}_{33,Y} - \langle \hat{\tau}_{33,Y}\rangle\}\,\mathrm{d}A \\ &= \epsilon^2 \hat{\Omega}_{3\beta} \iint_D X_\alpha \bar{\tau}_{\beta 3}\,\mathrm{d}A + \epsilon^2 \frac{\mathrm{d}}{\mathrm{d}Y} \iint_D X_\alpha \bar{\tau}_{33}\,\mathrm{d}A + O(\epsilon^3) \\ &= \frac{1}{2}\epsilon^2 \hat{\Omega}_{3\beta} e_{\alpha\beta} \bar{K}_3 + \epsilon^2 e_{\beta\alpha} \frac{\mathrm{d}\bar{K}_\beta}{\mathrm{d}Y} + O(\epsilon^3), \end{aligned}$$

because $\bar{\tau}_{\beta 3} = \bar{\tau}_{3\beta} = e_{\beta\gamma}\theta_{3,\gamma}$ and

$$\bar{K}_3 = -\iint_D (X_1\theta_{3,1} + X_2\theta_{3,2})\,\mathrm{d}A = 2\iint_D \theta_3\,\mathrm{d}A.$$

Now, from (38) and (39) we have $\bar{q}_{\alpha\beta} = K_{\alpha\beta}(\hat{a} + \hat{\Omega}_{3\gamma}X_\gamma) + e_{\alpha\beta}\hat{\Omega}_{3\delta}e_{\gamma\delta}K_{\epsilon\gamma}X_\epsilon$, and, since

$$\iint_D X_\alpha \bar{\tau}_{3\gamma}\,\mathrm{d}A = -e_{\gamma\alpha}\iint \theta_3\,\mathrm{d}A = \frac{1}{2} e_{\alpha\gamma}\bar{K}_3,$$

we may show that

$$\iint_D \bar{\tau}_{3\beta}\bar{q}_{\alpha\beta}\,\mathrm{d}A = 0.$$

Also, since $\bar{\tau}_{\alpha\beta} = 0$, the third and fourth terms in (42) reduce to

$$\begin{aligned}\epsilon^2 \iint_D (\bar{\tau}_{33}\bar{q}_{\alpha 3} - \bar{\tau}_{\alpha 3}\bar{q}_{33})\,\mathrm{d}A &= \epsilon^2 \hat{\Omega}_{\alpha\beta} \iint_D X_\beta \bar{\tau}_{33}\,\mathrm{d}A - \epsilon^2 \hat{\Omega}_{3\beta} \iint_D X_\beta \bar{\tau}_{\alpha 3}\,\mathrm{d}A \\ &= \epsilon^2 \hat{\Omega}_{\alpha\beta} e_{\gamma\beta} \bar{K}_\gamma - \frac{1}{2}\epsilon^2 \hat{\Omega}_{3\beta} e_{\beta\alpha} \bar{K}_3. \end{aligned}$$

Inserting the above results into (42) we obtain

$$\epsilon\hat{F}_\alpha = e_{\beta\alpha}\left(\epsilon M^*_\beta - \epsilon^2 \hat{\Omega}_{3\beta}\bar{K}_3 + \epsilon^2 \frac{\mathrm{d}\bar{K}_\beta}{\mathrm{d}Y}\right) + \epsilon^2 \hat{\Omega}_{\alpha\beta} e_{\gamma\beta} \bar{K}_\gamma + O(\epsilon^3). \tag{43}$$

In expressions (43) for the shear resultants, the approximate stress-couples are related to $\hat{\Omega}$ by the formulae

$$\bar{K}_3 = 2\iint_D \theta_3 \, \mathrm{d}A = \gamma \iint_D 2\psi \, \mathrm{d}A$$

$$\bar{K}_\alpha = e_{\alpha\beta}\hat{\Omega}_{3\gamma} \iint_D X_\beta X_\gamma \, \mathrm{d}A \equiv A\hat{\Omega}_{3\gamma} e_{\alpha\beta} I_{\beta\gamma} \tag{44}$$

and so are calculated on the basis of the St.-Venant semi-inverse solutions. They involve the usual torsional and bending rigidities and the familiar moments and products of inertia $I_{\beta\gamma}$ of the cross section D. The only influence of anisotropy occurs in the coefficients of (40), and hence in the torsional rigidity.

Similar manipulations may be applied to eqn (19) giving

$$\begin{aligned}\epsilon M_3^* &= \epsilon^2 e_{\alpha\beta} \iint_D \{\mathcal{N}_{\beta\alpha} - X_\alpha(\hat{\Omega}_{\beta j}\hat{\tau}_{j3} + \hat{\tau}_{\beta 3,Y})\} \, \mathrm{d}A \\ &= -\epsilon^2 e_{\alpha\beta} \iint_D \{\hat{\Omega}_{\beta j} X_\alpha \bar{\tau}_{j3} + X_\alpha \bar{\tau}_{\beta 3,Y} + O(\epsilon)\} \, \mathrm{d}A \\ &= \epsilon^2 \hat{\Omega}_{\beta 3}\bar{K}_\beta - \epsilon^2 \frac{\mathrm{d}\bar{K}_3}{\mathrm{d}Y} + O(\epsilon^3),\end{aligned} \tag{45}$$

where we have used the symmetry of c_{ijkl} and the result that

$$\begin{aligned} e_{\alpha\beta} \iint_D \mathcal{N}_{\beta\alpha} \, \mathrm{d}A &= -e_{\alpha\beta} \iint_D \{\bar{\tau}_{\alpha j}\bar{q}_{\beta j} + O(\epsilon)\} \, \mathrm{d}A \\ &= -e_{\alpha\beta} \iint_D \{\bar{\tau}_{\alpha 3}\hat{\Omega}_{\beta\gamma} X_\gamma + O(\epsilon)\} \, \mathrm{d}A = O(\epsilon). \end{aligned}$$

Equations (43) and (45) may be combined into the single equation

$$\epsilon^2 \frac{\mathrm{d}\bar{K}_i}{\mathrm{d}Y} + \epsilon^2 \hat{\Omega}_{ij}\bar{K}_j + \epsilon M_i^* + \epsilon e_{\alpha i 3}\hat{F}_\alpha = O(\epsilon^3). \tag{46}$$

We now revert to physical variables (on the scale of X_i) defined by

$$F_i^+ \equiv EF_i = \epsilon E\hat{F}_i, \qquad K_i^+ \equiv \epsilon E\bar{K}_i \simeq \epsilon E\hat{K}_i = EK_i$$

$$L_i^+ \equiv \epsilon EL_i, \qquad M_i^+ \equiv \epsilon EM_i^* \simeq \epsilon E\hat{M}_i = EM_i$$

and similarly rescale the curvatures and the torsion by introducing the components $\alpha^+ = \epsilon\alpha$, $\beta^+ = \epsilon\beta$, $\gamma^+ = \epsilon\gamma$ of the skew symmetric matrix $\mathbf{\Omega} \equiv \epsilon\hat{\mathbf{\Omega}}$. Then, by approximating (46) we obtain, using (5), (25), (26) and (44), the complete set of equations

$$\frac{\mathrm{d}F_i^+}{\mathrm{d}X_3} + \Omega_{ij}F_j^+ + L_i^+ = 0 \tag{47}$$

$$\frac{\mathrm{d}K_i^+}{\mathrm{d}X_3} + \Omega_{ij}K_j^+ + M_i^+ + e_{\alpha i 3}F_\alpha^+ = 0 \tag{48}$$

$$\frac{\mathrm{d}H_{ij}}{\mathrm{d}X_3} = H_{ik}\Omega_{kj}, \qquad \mathbf{H}\mathbf{H}^T = \mathbf{I}$$

$$K_\alpha^+ = EA\Omega_{3\gamma}e_{\alpha\beta}I_{\beta\gamma}, \qquad K_3^+ = 2E\gamma^+ \iint_D \psi \, \mathrm{d}A \tag{50}$$

$$\mathbf{\Omega} = \begin{pmatrix} 0 & -\gamma^+ & \beta^+ \\ \gamma^+ & 0 & -\alpha^+ \\ -\beta^+ & \alpha^+ & 0 \end{pmatrix}$$

which, apart from a change of notation, are the equations to be found on p. 388 of Love[1]. They are the equations for the equilibrium of an *elastica* and are simplified when the X_1 and X_2 axes are chosen as principal inertial axes of D, so that $I_{12} = I_{21} = 0$. Methods for their solution in many situations may be found in [1] and [3].

The elastica equations are now seen to be logical approximations to three dimensional elasticity for slender rods, provided that the *strains* are small enough for a linear elastic constitutive law to hold and that external forces are neither so large nor so rapidly varying that the curvatures and the torsion vary significantly over lengths comparable with the thickness of the rod. For any material having the symmetry discussed in Section 6, the bending rigidities are correctly approximated by the assumptions that normal cross sections remain normal to the curve of centres and are undistorted, whilst the torsional rigidity is correctly given in (50) by solving a standard torsion problem for ψ. The couple resultant $\mathbf{M}^+$ is also correctly approximated by neglecting distortions (but not rigid rotations) of each cross section. When the elastica equations are solved for $\mathbf{\Omega}$, $\mathbf{K}^+$, $\mathbf{F}^+$ and $\mathbf{H}$ (which must be gradually varying functions of X_3), the extension is found from $\epsilon\hat{a} = \epsilon\bar{F}_3/A \simeq F_3^+/EA$. Generally this is small, so that the standard assumption that the elastica is inextensible is justified, and (21) may be replaced by

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}X_3} \simeq \mathbf{e}(\epsilon X_3).$$

The actual distortions of each cross section are those of type (37) corresponding to the local values of the curvatures and the torsion—the extension effects usually being negligible. Examples showing how standard small displacement solutions may be readily generalized to situations involving large displacements and rotations will be found in [12].

8. THE END

The results in Section 7 justify the use of the elastica equations (and their various approximations for beam bending, column buckling, etc.) for any cross-sectional shape and a wide range of elastic materials and loading conditions. The fact that details of the end loading cannot be taken into account is usually explained by an appeal to St.-Venant's principle that loading details over a portion of the boundary surface becomes unimportant at significant distances from that portion. Since we have shown how the rod description arises naturally from a singular perturbation procedure, it is hardly suprising that a search for a "boundary layer" or "transition" solution leads naturally to such a statement.

Near $X_3 = 0$, L the deformation depends on X_3 as well as on X_α and $Y = \epsilon X_3$. Consequently, near $X_3 = 0$ we amend (4) and seek a configuration written in inner variables $\mathbf{X}$ as

$$\mathbf{x} = \mathbf{r}(X_3) + \mathbf{H}(\epsilon X_3)\mathbf{u}(\mathbf{X}), \qquad \mathbf{u}(\mathbf{X}) = \mathbf{u}^*(X_\alpha, \epsilon X_3) + \epsilon\hat{\mathbf{u}}(\mathbf{X}).$$

Then, with $u_i^* = X_\alpha\delta_{i\alpha}$ as in Sections 5–7, (24) is modified slightly to become

$$\hat{q}_{i\alpha} = \hat{u}_{i,\alpha}, \quad \hat{q}_{i3} = \hat{a}(\epsilon X_3)\delta_{i3} + \hat{\Omega}_{i\beta}(\epsilon X_3)X_\beta + \hat{u}_{i,3} + \epsilon\hat{\Omega}_{ij}\hat{u}_j. \tag{51}$$

The stress $\epsilon\hat{\tau}$ also depends on X_3 so that (10) is replaced by

$$\frac{\partial\hat{\tau}_{ij}}{\partial X_j} + \epsilon\hat{\Omega}_{ij}(\epsilon X_3)\hat{\tau}_{j3} = -H_{ji}\hat{f}_j \quad \text{in} \quad D \times (0, \epsilon^{-1}L) \tag{52}$$

whilst the boundary condition (11) over $\partial D \times (0, \epsilon^{-1}L)$ is essentially unchanged. Since $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$ are $O(\epsilon)$, the limiting form of equations (51), (52) and (11) is then

$$\frac{\partial\hat{\tau}_{ij}}{\partial X_j} = 0 \quad \text{in} \quad D \times (0, \infty), \quad \hat{\tau}_{i\alpha}N_\alpha = 0 \quad \text{over} \quad \partial D \times (0, \infty)$$

$$\hat{\tau}_{ij} = c_{ijkl}\hat{q}_{kl}, \quad \hat{q}_{i\alpha} = \hat{u}_{i,\alpha}, \quad \hat{q}_{i3} = \hat{a}(0)\delta_{i3} + \hat{\Omega}_{i\beta}(0)X_\beta + \hat{u}_{i,3}.$$

Solutions to these may be written as the sum of two parts

$$\hat{\mathbf{u}} = \bar{\mathbf{u}}(X_\alpha, 0) + \mathbf{u}^\epsilon(\mathbf{X}),$$

where $\bar{\mathbf{u}}(X_\alpha, 0)$ is the combination (37) of St.-Venant semi-inverse solutions appropriate to the stretching, bending and torsion at $Y = 0$. Then $\mathbf{u}^\epsilon$ satisfies

$$\tau^\epsilon_{ij,j} = 0 \quad \text{in} \quad D \times (0, \infty), \qquad \tau^\epsilon_{i\alpha} N_\alpha = 0 \quad \text{over} \quad \partial D \times (0, \infty), \qquad \tau^\epsilon_{ij} = c_{ijkl} u^\epsilon_{k,l}$$

and will be a first approximation to the end correction at $X_3 = 0$ if we insist that $\mathbf{u}^\epsilon \to \mathbf{0}$, $\boldsymbol{\tau}^\epsilon \to \mathbf{0}$ as $X_3 \to \infty$ and we impose suitable conditions over $X_3 = 0$.

Suppose, for definiteness, that the tractions are specified over $X_3 = 0$ as $\hat{\tau}_{i3} = \sigma_i(X_\alpha)$, then we have

$$\tau^\epsilon_{i3}(X_\alpha, 0) = \sigma_i(X_\alpha) - c_{i3k\alpha} \bar{u}_{k,\alpha} - c_{i3k3}\{\hat{a}(0)\delta_{k3} + \hat{\Omega}_{k\beta}(0) X_\beta\}.$$

Toupin[11] in his analysis of St.-Venant's hypothesis, analyses configurations in which $u_i^\epsilon \tau^\epsilon_{i3} \to 0$ as $X_3 \to \infty$ and shows that the associated strain energy in $X_3 > l$ decays exponentially as $l \to \infty$. These configurations involve an arbitrary rigid body translation and rotation, which may be used to set $\mathbf{u}^\epsilon \to \mathbf{0}$ as $X_3 \to \infty$. Then $\mathbf{u}^\epsilon(X_\alpha, 0)$ is uniquely determined, and it can be shown that the pointwise decay of $\mathbf{u}^\epsilon$ with X_3 is exponential, with decay rate depending on c_{ijkl} and related to the lowest frequency of free vibration of a disc having shape D. The decay rate clearly is independent of ϵ.

Knowing $\mathbf{u}^\epsilon(X_\alpha, 0)$ we may compute any $O(\epsilon)$ corrections to the rotation $\mathbf{H}(0)$ and displacement $\mathbf{r}(0)$ of the end. Moreover, since $\boldsymbol{\tau}^\epsilon$ is a stress-distribution corresponding to vanishing tractions over $\partial D \times (0, \infty)$, we must have

$$\iint_D \tau^\epsilon_{i3}(X_\alpha, 0)\, \mathrm{d}A = 0, \qquad e_{imn} \iint_D X_m \tau^\epsilon_{n3}(X_\alpha, 0)\, \mathrm{d}A = 0.$$

Thus, to the approximations implied by (47)–(50), the stress-couple $K_i^+ = \epsilon E \bar{K}_i$ at $Y = 0$ must equal the moment of the tractions applied over the end.

Note added in proof. The author has recently become aware that Rigolot [13] has developed a similar asymptotic description for large displacements of slender rods within second-order elasticity theory. Also, in [14], he uses matching of asymptotic expansions to discuss end effects occurring in a small displacement theory.

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